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Invariant densities of delayed maps in the limit of large time delay

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The marginal invariant density of chaotic attractors of scalar systems with time delayed feedback has an asymptotic form in the limit of large delay. It is well known that the dimension and the entropy of such attractors obey interesting scaling laws in this limit, but very little has been said about properties of the invariant density. We present general considerations, detailed analytical results in low order perturbation theory for a particular model, and numerics for understanding the asymptotic behavior of the projections of the invariant density. Our approach clarifies how the analytical properties of the model determine the behavior of the marginal invariant densities for large delay times.

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I. INTRODUCTION

Dynamical systems with a retarded feedback appear in many different situations in nature and technology like physiology [1,2], biology [3], laser physics [4], and economy [5]. Such systems are usually modeled by delayed differential equations (DDE's) of the form

$$\dot{x}=F(x(t),x(t-\tau)). \tag{1}$$

Apart from its relevance to applied sciences, the system (1) also has interesting theoretical features: The phase space of Eq. (1) is infinite dimensional and high dimensional chaotic attractors may appear [6]. This is indeed a nice example of a simple system that can show high dimensional chaotic attractors.

The dynamical behavior of a given model of type (1) depends, for small values of τ, typically in a very complicated way on τ. However, there is ample evidence that many models enter, for sufficiently large τ, in a regime where the τ dependence becomes very simple. In particular, the Kaplan-Yorke dimension of these chaotic attractors scales linearly with the delay value, and the information entropy estimated with the Pesin identity achieves a finite asymptotic value for large τ [7,8]. In fact, these scaling properties are a consequence of the asymptotic behavior of the Lyapunov spectrum in the limit $\tau\rightarrow\infty$. Being ergodic averages, Lyapunov exponents reflect two important aspects of the dynamics: the linear (instability) and the statistical properties. The latter depend directly on the invariant density of a system. In order to gain some insight in this "universal" regime of high dimensional chaos of delayed systems, the understanding of the properties of invariant densities of delayed systems in the limit of large delay is an essential and nontrivial starting point.

Another well known source of high dimensional chaos are spatially extended systems. There are different mechanisms leading to high dimensional chaos in spatially extended systems and in systems with time delayed feedback, but for several scenarios it was possible to establish a formal relation between both as in [9,10] or in [11]. In these cases, the limit $\tau\rightarrow\infty$ was identified with the thermodynamic limit of extended systems, which again emphasizes our interest in the large delay limit. The marginal invariant density $\rho(x)$ of a time delayed system then corresponds to the single-variable density of a spatially extended system, where the existence of a well defined limit of the latter in the thermodynamic limit is well known [12] .

So, in this work we focus on time delayed feedback systems with chaotic attractors, in a regime of the feedback time τ where the above-mentioned asymptotics of the Kaplan-Yorke dimension and of the information entropy can be observed. By ergodicity, a single solution $x(t)$ then creates the natural invariant density called $\rho_\tau(x)$, if transients are discarded. This marginal density is one particular projection of the full phase space density. The main issue of this paper is to study under which conditions and how this density $\rho_\tau(x)$ converges to an asymptotic form $\rho_\infty(x)$ in the limit of large τ (see Fig. 1 for an example where an asymptotic form exists in a range of delay values), and what are the underlying mechanisms for this convergence.

One could argue that this behavior is not surprising: As the dimensions of the attractors grow and more degrees of freedom become relevant one could expect that the projection of the measures onto any space with a much smaller dimension than the attractor itself will be smooth and not depend on the delay. This would be a consequence of the central limit theorem. But this is far from being the correct explanation: The degrees of freedom are correlated and the one-dimensional distribution is typically not Gaussian (as the argument would predict) but strongly dependent on the system. The idea of using the central limit theorem was, however, explored in [13]. The authors are able to identify the shortest time scale of correlations and construct a version of the central limit theorem to derive the invariant density of uncorrelated points. But this situation is not general and one can even find systems with low-dimensional attractors ($d\approx3$) where invariance of the density with respect to the delay value is observed.

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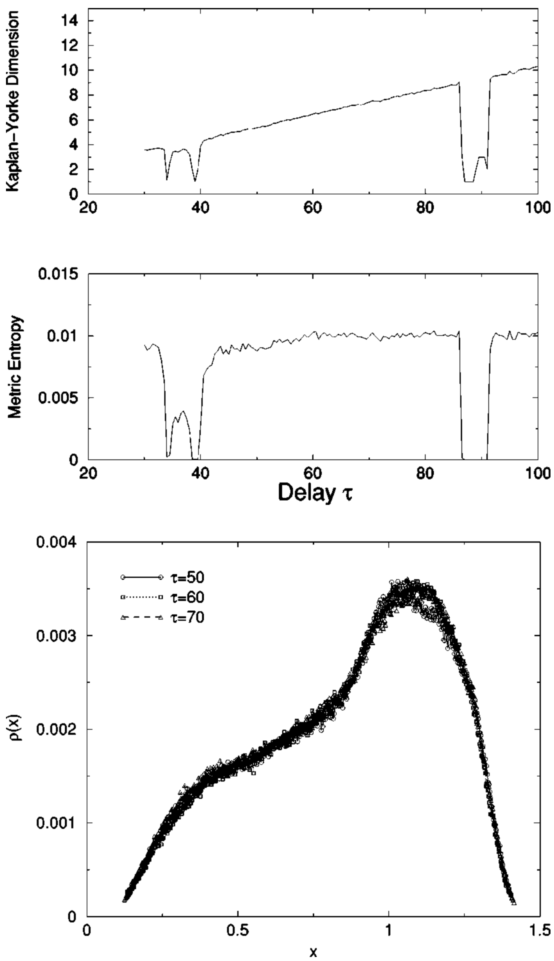

FIG. 1. Some properties of the Mackey-Glass equation $\dot{x} = -bx(t) + ax(t-\tau)/[(1+x(t-\tau)^{10})]$ with $a=0.2$ and $b=0.1$. Upper panel: dimension, entropy as a function of the delay τ. Lower panel: probability densities constructed from the time series of $x(n\Delta t)$ with $\Delta t = 0.001 * \tau$ (integration step).

As a side remark, we want to mention that not all systems of type (1) show the above-mentioned scenario. In particular, two different types of behaviors have been found as being typical as well, namely multistability [14] and asymptotic periodicity [11,15]. In both situations the convergence properties of the Frobenius-Perron operator induced by the dynamics will depend strongly on the initial density and therefore the system is not ergodic [16], so that (quasi-)periodic solutions of the Frobenius-Perron equation exist. In these cases there exist also fixed point solutions corresponding to the natural invariant density on a single ergodic component, whose dependence on τ could be studied with the concepts of this paper, but it would involve additional complications when we have to relate a particular invariant set for one τ value to a particular one for another τ value. Hence, we restrict our investigation to systems with a globally attracting chaotic invariant set.

The concrete treatment of problems related to the invariant density of Eq. (1) will at some point require discretization of time as done in [11,13]. Therefore, one will be treating the Frobenius-Perron operator of a map that in some limit will describe very well the behavior of the continuous time system. Since we are interested in fundamental issues and not in the behavior of a special system, we decided to treat directly scalar delayed maps of the form

$$x_{n+1} = f(x_n, x_{n-\tau}), \tag{2}$$

for which we have observed that the probability density $\rho(x)$ of x_n induced by a stationary invariant density in the finite $\tau+1$-dimensional phase space will have an asymptotic form for large delay. Therefore, in this sense delayed maps and DDE's show similar behaviors.

The limit $\tau \to \infty$ for maps like Eq. (2) can be turned into a continuum limit which is equivalent to the discretization of DDE's. A good approximation of a discretized DDE for the true DDE's solutions is only obtained in this limit [11]. Therefore, one could say that some delayed maps in the limit of large delay approximate the solutions of delayed differential equations, so that one expects in fact that in this limit their invariant density should assume an asymptotic form invariant with respect to the delay.

In Sec. II we describe the open problems related to the invariant densities of delayed maps and illustrate them with numerical results obtained for a special map. In Sec. III we analyze a simple case analytically. Our approach consists of investigating a very simple delayed map: a shift on a torus with a weak time delayed periodic perturbation. All the calculations are performed in the Fourier space and the Fourier coefficients are calculated up to first order. Using this approach we are able to obtain explicitly the asymptotic form of the projections of the invariant density in the limit of large delays and analyze its convergence as the delay increases. Finally, in Sec. IV we present a discussion of the results.

II. NUMERICAL OBSERVATIONS AND STATEMENT OF THE PROBLEM

In many different delayed systems we have observed an asymptotic behavior of the projections of the invariant measure at large delays. As an example consider the map

$$x_{n+1} = (1-\epsilon) f(x_n) + \epsilon f(x_{n-\tau}), \tag{3}$$

where $f(x) = 2x - \mathrm{sgn}(x)$, $x \in [-1,1]$. In Fig. 2 we present the numerical results on how the density of the variable x converges to an asymptotic form as the delay increases. The densities were obtained by dividing the interval $[-1,1]$ in cells I_x of equal size centered at a point x. The density $\mu(x)$ is computed from a normalized histogram (relative visiting frequency of cell I_x). We have defined a quantity to characterize the difference between these invariant densities at low and large delay: $\Sigma_{I_x} |\mu_\tau(x) - \mu_\infty(x)|$. Its dependency with the delay value is depicted in Fig. 3. This quantity converges to zero (at least within the numerical error) as $\tau \to \infty$. The convergence behavior depends on details of the system (here the parameter ϵ). In the special case of $\epsilon = 0.5$, the measure has the form $\mu(x) \propto 2x - \mathrm{sgn}(x)$ and independent of τ. As the densities are non-Gaussian, we do not expect that the simple

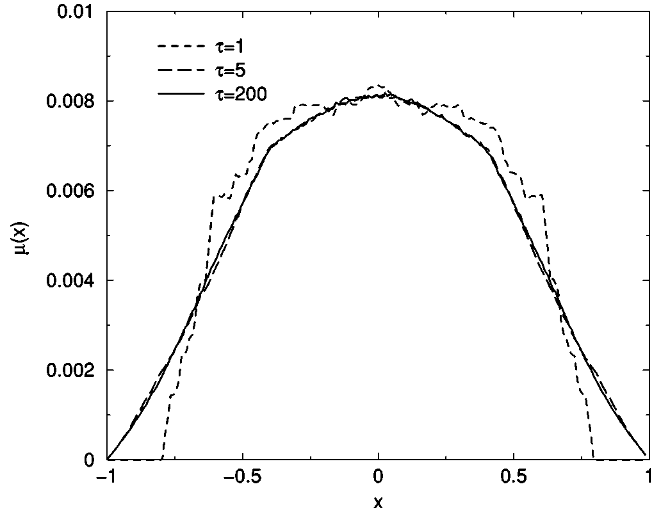

FIG. 2. The invariant density of Eq. (3) for a definite τ and $\epsilon=0.3$. $\mu(x)$ is estimated by a normalized histogram. The interval $[-1,1]$ is divided into 200 cells and we have used a time series of 10^6 points.

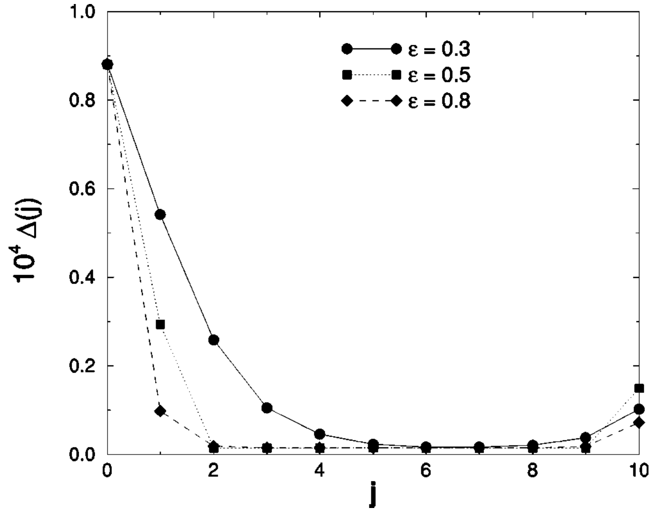

FIG. 4. $\Delta(x_n, x_{n-j})$ for the map (3) with delay $\tau=10$. The measures are estimated from the relative frequencies at cells on a plane and the average is performed over the cells. A time series of 10^7 point is used.

central limit theorem will supply justification for the convergence. In fact, as it will be seen later, the variables are not completely uncorrelated.

In order to observe if there is some decoupling of the degrees of freedom we define a quantity

$$\Delta(j)=\langle|\mu(x_n,x_{n-j})-\mu(x_n)\mu(x_{n-j})|\rangle, \tag{4}$$

where the two-dimensional density $\mu(x_n,x_{n-j})$ is estimated by dividing the plane in square cells centered at $\{x_n,x_{n-j}\}$ and constructing the corresponding histogram from a time series. The average in Eq. (4) is computed over the cells. This quantity has similar meaning as the mutual information: It describes the distance between two densities. The nearer the quantity is to zero, the more uncorrelated are x_n and x_{n-j}. This is a stronger test for uncorrelation than the linear correlation would provide. In Fig. 4 the dependency of Δ on j is depicted for different values of ϵ. Due to statistical errors existent on the measure at every box, the value $\Delta=0$ is never achieved. We could instead identify a finite minimum value for Δ at every simulation (the plateau in Fig. 4). We consider that the variables are uncorrelated when Δ assumes this minimum value. By comparing the Figs. 3 and 4 one might wonder if the convergence of the one-dimensional projection of the measure is a consequence of the loss of correlations at short time scales.

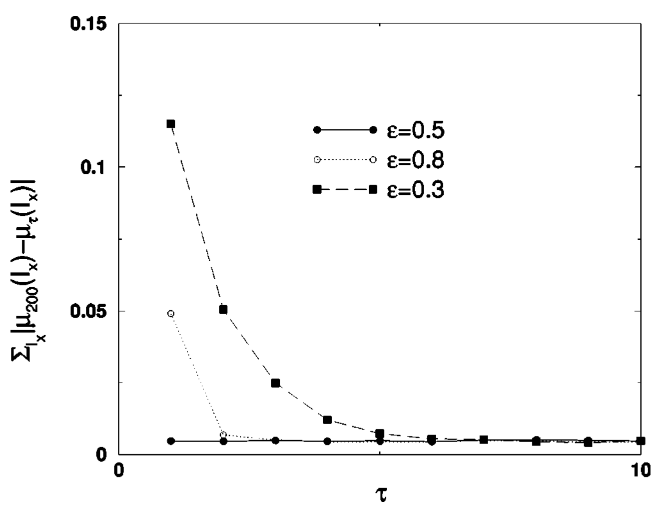

FIG. 3. Difference between the measures μ_τ and μ_{200} (representing μ_∞) as a function of τ. The measure is estimated in the same way as in Fig. 2. The sum is performed over the cells as described in the text.

In order to investigate formally the invariant measures, we have to construct the Frobenius-Perron equation and try to understand the observed numerical facts from them. There are two different approaches to construct these equations.

One way is to consider the τ-time distributions

$$\begin{aligned}\rho^{(1)}(x)&=\langle\delta(x-x_n)\rangle,\\ \rho^{(2)}(x,y)&=\langle\delta(x-x_n)\,\delta(y-x_{n-\tau})\rangle,\\ \rho^{(3)}(x,y,z)&=\langle\delta(x-x_n)\,\delta(y-x_{n-\tau})\,\delta(z-x_{n-2\tau})\rangle,\ \dots,\end{aligned} \tag{5}$$

where $\langle\cdots\rangle$ denotes the average over x_n with respect to the natural invariant density (i.e., a long time average), then invariance yields the system of equations

$$\begin{aligned}\rho^{(1)}(x)&=\int dx'\int dy'\,\delta(x-(1-\varepsilon)f(x')-\varepsilon f(y'))\rho^{(2)}(x',y'),\\ \rho^{(2)}(x,y)&=\int dx'\int dy'\int dz'\,\delta(x-(1-\varepsilon)f(x')-\varepsilon f(y'))\\ &\quad\times\delta(y-(1-\varepsilon)f(y')-\varepsilon f(z'))\rho^{(3)}(x',y',z'),\\ &\ \ \vdots\end{aligned} \tag{6}$$

which corresponds to an open hierarchy of equations that cannot be solved unless some simple ansatz is assumed, for instance assuming that $\rho^{(2)}(x,y)=\rho^{(1)}(x)\rho^{(1)}(y)$. An interesting feature of the system (6) is that it does not depend explicitly on the delay value, while one expects that the invariant density does. The dependency comes implicitly in the fact that correlations, e.g., between x and y, would depend on the delay value.

Another way to look at the problem is to consider the delayed map (3) acting on a vector space of dimension $\tau+1$. Defining the components of the vector by the notation $x^{(j)}=x_{n-j}$, the Frobenius-Perron equation in terms of these coordinates reads

$$\rho(x^{(0)},\ldots,x^{(\tau)})=\int dz\,\delta(x^{(0)}-(1-\varepsilon)f(x^{(1)})-\varepsilon f(z))\rho(x^{(1)},\ldots,x^{(\tau)},z). \tag{7}$$

Its solution determines the two time density

$$\rho^{(2)}(x,y)=\int dx^{(1)}\cdots\int dx^{(\tau-1)}\rho(x,x^{(1)},\ldots,x^{(\tau-1)},y). \tag{8}$$

Hence, one quantity in the system (6) is fixed, and by condition (8) the delay time enters explicitly. Therefore, an analysis which is based solely on Eqs. (6) does not seem to be consistent.

Considering Eqs. (6), we see that $\rho^{(1)}$ is fully determined if $\rho^{(2)}$ is known. If $\rho^{(2)}$ has a definite asymptotic form in the limit $\tau\to\infty$ so does $\rho^{(1)}$, according to Eq. (6). Determining $\rho^{(2)}$ seems to be only possible by solving the Frobenius-Perron equation (7), which is a difficult task particularly at the limit of large τ. In order to investigate this problem further we have chosen a special case of a delayed system where the invariant measure can be investigated analytically.

III. SHIFTS ON A TORUS WITH A DELAYED PERTURBATION

In order to perform some analytical investigations we have chosen a map on a torus, i.e., we consider its variable φ as an angle. Such maps are known to have nice properties from the analytical point of view, e.g., they are hyperbolic if local expansion rates are positive and allow for perturbation expansions (cf., e.g., [17] for an application in the context of coupled map lattices). Since we will base part of our analysis on such expansions we consider the following map defined on the circle:

$$\varphi_{n+1}=2\varphi_n+\varepsilon g(\varphi_n)+\varepsilon g(\varphi_{n-\tau}), \tag{9}$$

where the variable φ is considered modulo 2π and ε will later on be a small parameter giving rise to a perturbation theory. We may also express this map in a vector space considering, $\boldsymbol{\phi}$ as vector with components $\phi^{(0)},\phi^{(1)},\ldots,\phi^{(\tau)}$ where $\phi_n^{(i)}=\varphi_{n-i}$,

$$\phi_{n+1}^{(0)}=2\phi_n^{(0)}+\varepsilon g(\phi_n^{(0)})+\varepsilon g(\phi_n^{(\tau)})=f(\phi^{(0)},\phi^{(\tau)}),$$
$$\phi_{n+1}^{(i)}=\phi_n^{(i-1)},\quad 1\leqslant i\leqslant\tau \tag{10}$$

i.e., $\phi_{n+1}=F(\phi_n)$. We can write the Frobenius-Perron equation for this system in this $\tau+1$-dimensional space as

$$\rho_{n+1}(\boldsymbol{\phi})=\int d\boldsymbol{\phi}'\,\delta(\boldsymbol{\phi}-F(\boldsymbol{\phi}'\eta))\rho_n(\boldsymbol{\phi}'). \tag{11}$$

Switching now to the Fourier decomposition

$$\rho_n(\boldsymbol{\phi})=\sum_{\mathbf{k}} c_n(\mathbf{k})e^{i\mathbf{k}\cdot\boldsymbol{\phi}},$$
$$c_n(\mathbf{k})=\frac{1}{(2\pi)^\tau}\int d\phi\,e^{-i\mathbf{k}\cdot\boldsymbol{\phi}}\rho_n(\boldsymbol{\phi}), \tag{12}$$

Eq. (11) reads

$$c_{n+1}(\mathbf{k})=\sum_{\mathbf{k}'} L_{\mathbf{k},\mathbf{k}'}c_n(\mathbf{k}'), \tag{13}$$

where

$$L_{\mathbf{k},\mathbf{k}'}=\Gamma(k'^{(0)}-2k^{(0)}-k^{(1)},k^{(0)})\Gamma(k'^{(\tau)},k^{(0)})\times\prod_{j=1}^{\tau-1}\delta_{k'^{(j)},k^{(j+1)}} \tag{14}$$

and the abbreviation

$$\Gamma(k',k)=\frac{1}{2\pi}\int d\phi\,e^{ik'\phi-ik\varepsilon g(\phi)} \tag{15}$$

has been introduced taking the delay into account. Expansion in terms of ε yields

$$\Gamma(k',k)=\delta_{k',0}-ik\varepsilon G_{k'}+O(\varepsilon^2), \tag{16}$$

where G_k' are the Fourier coefficients of g.

Evaluating Eq. (13) for $\varepsilon=0$ we have

$$c_{n+1}(k^{(0)},k^{(1)},k^{(3)},\ldots,k^{(\tau-1)},k^{(\tau)})=c_n(2k^{(0)}+k^{(1)},k^{(2)},k^{(3)},\ldots,k^{(\tau)},0) \tag{17}$$

and thus

$$c_{n+\tau}(k^{(0)},k^{(1)},k^{(3)},\ldots,k^{(\tau-1)},k^{(\tau)})=c_n(N_\tau(\mathbf{k}),0,\ldots,0,0,0), \tag{18}$$

where the notation

$$N_\nu(\mathbf{k})=2^\nu k^{(0)}+2^{\nu-1}k^{(1)}+\cdots+k^{(\nu)} \tag{19}$$

for the argument of the Fourier coefficients has been used. If we consider an analytic density at time $n=0$ then its Fourier coefficients decay exponentially. Thus, iterating the system

(17) we recognize that all coefficients but a few become exponentially small and we end up with the stationary solution

$$c_*^{(0)}(\mathbf{k}) = \delta_{N(\mathbf{k}),0}. \tag{20}$$

The form of the invariant density in the whole phase space is the following:

$$\rho(\boldsymbol{\phi})|_{\varepsilon=0} = \sum_{\mathbf{k}} \delta_{N(\mathbf{k}),0} e^{i\mathbf{k}\cdot\boldsymbol{\phi}} \tag{21}$$

and it consists of one-dimensional strips with uniform density. The projection of this invariant density on one dimension is uniform. The picture of the strips can be seen easily in two dimensions. For instance, considering the projections on the planes $(\phi^{(0)},\phi^{(j)})$ one has

$$\rho^{(0)}(\phi^{(0)},\phi^{(j)})|_{\varepsilon=0} = \delta(\phi^{(0)} - 2^{-j}\phi^{(j)}) \tag{22}$$

and for large j the strips practically fill the plane. These results of course follow from analyzing the map without the delayed term.

We use the series expansion

$$c_n(\mathbf{k}) = c_n^{(0)}(\mathbf{k}) + \varepsilon c_n^{(1)}(\mathbf{k}) + O(\varepsilon^2) \tag{23}$$

to determine the stationary solution for nonvanishing ε. Combining Eqs. (13), (16), and (23) we obtain

$$\begin{aligned} c_{n+1}^{(1)}(\mathbf{k}) = {} & c_n^{(1)}(2k^{(0)}+k^{(1)},k^{(2)},\ldots,k^{(\tau)},0) \\ & - ik^{(0)} \sum_{\mathbf{k}'} [\delta_{k'^{(\tau)},0}\delta_{N(\mathbf{k}'),0} G_{k'^{(0)}-k^{(1)}-2k^{(0)}} \\ & + G_{k'^{(\tau)}}\delta_{N(\mathbf{k}'),0}\delta_{k'^{(0)},2k^{(0)}+k^{(1)}}] \prod_{j=1}^{\tau-1} \delta_{k'^{(j)},k^{(j+1)}}. \end{aligned} \tag{24}$$

Using similar arguments as before, we obtain a stationary solution for the first order coefficients that reads

$$\begin{aligned} c_*^{(1)}(\mathbf{k}) = {} & -i\sum_{\nu=0}^{\infty} 2^{\nu} N_\tau(\mathbf{k}) \\ & \times \sum_{\mathbf{k}'} [\delta_{k'^{(\tau)},0}\delta_{N_\tau(\mathbf{k}'),0} G_{k'^{(0)}-2k^{(0)}-k^{(1)}} \\ & + G_{k'^{(\tau)}}\delta_{N_\tau(\mathbf{k}'),0}\delta_{k'^{(0)},2k^{(0)}+k^{(1)}}] \prod_{j=1}^{\tau-1} \delta_{k'^{(j)},0} \\ & - i\sum_{\nu=0}^{\tau-1} N_\nu(\mathbf{k}) \sum_{\mathbf{k}'} [\delta_{k'^{(\tau)},0}\delta_{N(\mathbf{k}'),0} G_{k'^{(0)}-2k^{(0)}-k^{(1)}} \\ & + G_{k'^{(\tau)}}\delta_{N(\mathbf{k}'),0}\delta_{k'^{(0)},2k^{(0)}+k^{(1)}}] \\ & \times \prod_{j=1}^{\tau-\nu-1} \delta_{k'^{(j)},k^{j+\nu+1}} \prod_{j=\tau-\nu}^{\tau-1} \delta_{k'^{(j)},0}. \end{aligned} \tag{25}$$

Now, we have an approximation for the invariant density up to first order. We are interested in the behavior of the low-dimensional projections of this invariant density and their dependence on the delay time τ. Let us first consider the one variable distribution: its expression is obtained considering $k^{(1)} = k^{(2)} = \cdots = k^{(\tau)} = 0$ in Eq. (12). We therefore make this substitution in Eq. (25) to obtain the form of the corresponding Fourier coefficients:

$$c(k^{(0)}) = \delta_{k^{(0)},0} + \varepsilon c^{(1)}(k^{(0)}) + O(\varepsilon^2), \tag{26}$$

with the contribution of first order being given by

$$\begin{aligned} c^{(1)}(k^{(0)}) = {} & -ik^{(0)} \sum_{\nu=0}^{\infty} 2^{\nu} G_{-2^{\nu+1}k^{(0)}} \\ & - ik^{(0)} \sum_{\nu=0}^{\infty} 2^{\nu} G_{-2^{\tau+\nu+1}k^{(0)}}. \end{aligned} \tag{27}$$

For $\varepsilon = 0$ the measure coincides with a uniform measure which is expected looking at Eq. (9). The delay-dependent part is contained in the higher order terms. From the first order approximation we can see that if g is a smooth function, this contribution will aquire an asymptotic form in the limit $\tau \to \infty$.

The same procedure can be applied to obtain two-dimensional projections. Considering for instance $k^{(i)} = 0$, for any $i \neq 0$ and $i \neq j$ we may obtain the coefficients

$$c(k^{(0)},k^{(j)}) = \delta_{2^\tau k^{(0)} + 2^{\tau-j}k^{(j)},0} + \varepsilon c^{(1)}(k^{(0)},k^{(j)}) + O(\varepsilon^2), \tag{28}$$

where

$$\begin{aligned} c^{(1)}(k^{(0)},k^{(j)}) = {} & -i(2^j k^{(0)} + k^{(j)}) \sum_{\nu=0}^{\infty} 2^{\nu} [G_{2^{\nu+1}(2^j k^{(0)}+k^{(j)})} \\ & + G_{2^{\tau+\nu+1}(2^j k^{(0)}+k^{(j)})}] - i2^{(j-1)} \\ & \times [G_{(2^j k^{(0)}+k^{(j)})} + G_{2^\tau(2^j k^{(0)}+k^{(j)})}] \\ & - ik^{(0)} \sum_{\nu=0}^{j-2} 2^{\nu} [G_{2^{\nu+1}(k^{(0)}+2^{-j}k^{(j)})}\delta_{k^{(j)},2^{j-\nu-1}m} \\ & + G_{-2^{\tau+\nu+1}(k^{(0)}+2^{-j}k^{(j)})}]. \end{aligned} \tag{29}$$

One should also expect that the two-dimensional projections converge towards an asymptotic form in the limit of large delay.

For ease of presentation and in order to clarify our ideas let us first consider a particular choice for the function g in Eq. (9), namely,

$$g(\varphi) = \varphi(\pi - \varphi) \quad \text{for} \quad \varphi \in [0,\pi], \qquad g(\varphi) = g(\varphi + \pi). \tag{30}$$

The Fourier coefficients of this function are

$$G_{2k} = \frac{-2}{(2k)^2}, \qquad G_{2k+1} = 0 \quad \forall \quad k. \tag{31}$$

With Eq. (31) in Eqs. (27) and (26), we have the first order approximation for the coefficients

$$c^{(1)}(k^{(0)}) = \frac{i}{k^{(0)}}\left(1 + \frac{1}{2^{2\tau}}\right) \tag{32}$$

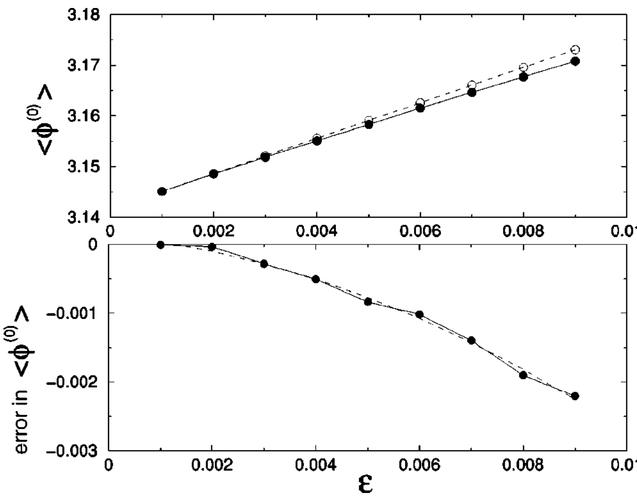

FIG. 5. Upper panel $\langle \phi^{(0)} \rangle$ as function of ε for $\tau=2$; open circles correspond to the first order approximation and the closed circles to numerics. Lower panel: the difference between first order and numerics; the dashed line is a fit to a parabola, showing that the error is of second order in ε.

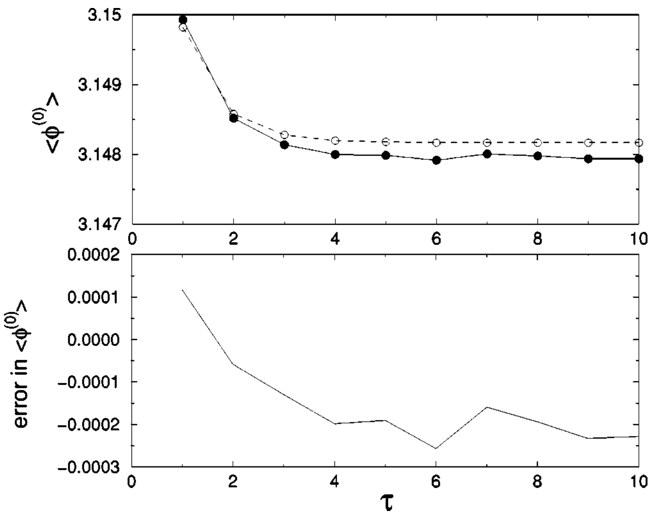

FIG. 7. Upper panel: $\langle \phi^{(0)} \rangle$ as a function of τ for $\varepsilon=0.002$; the open and closed circles have the same meaning as in previous figures. Lower panel: the difference between the first order approximation and numerics is depicted.

and for the one-dimensional projection of the invariant measure

$$\rho(\phi^{(0)})=1-2\varepsilon\left(1+\frac{1}{2^{2\tau}}\right)\sum_{k^{(0)}=1}^{\infty}\frac{\sin(k^{(0)}\phi^{(0)})}{k^{(0)}}+O(\varepsilon^{2}). \tag{33}$$

It is easy to see that this projection converges smoothly towards an asymptotic form. We can check numerically the range of validity of our approximation observing the behavior of the averages with respect to the invariant measure $\langle \phi^{(0)} \rangle, \langle \cos(\phi^{(0)}) \rangle, \langle \sin(\phi^{(0)}) \rangle$. In order to give the average of the angle a definite meaning we have restricted the values of the angle to the interval $[0,2\pi)$. In Figs. 5 and 6 some numerical results are compared to the first order approximation as a function of the coupling ϵ. All the numerical results clearly confirm that our approximation correctly describes the lowest order effects, and that the neglected higher order corrections lead to less than 10% effects for $\epsilon<0.01$. In order to demonstrate that our results are to some extent uniform in the delay time we investigate the error for finite coupling ϵ in dependence on the delay time. The result displayed in Fig. 7 shows that the difference stays finite even if the delay becomes large. Such finding demonstrates that our expansion can be used even for large delay times.

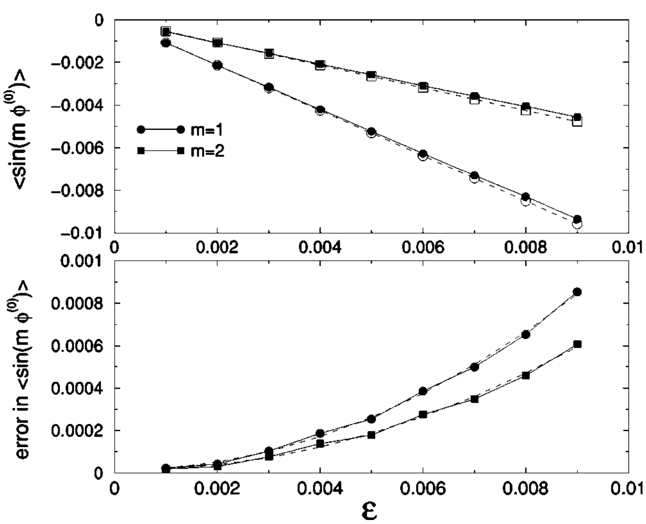

FIG. 6. Similar to Fig. 5, with averages of $\langle \sin(m\phi^{(0)}) \rangle$, which are related to the values of the coefficients $c(m)$. In the upper panel the closed symbols correspond to the first order approximation and the open ones to numerics. In the lower panel the difference between the first order approximation and the numerics is depicted. The fit to a parabola shows that the error is of second order.

Also the specific expressions for the two-dimensional projections may be obtained,

$$\rho(\phi^{(0)},\phi^{(j)})=\delta(\phi^{(0)}-2^{-j}\phi^{(j)}) +\epsilon\sum_{k^{(0)},k^{(j)}}c^{(1)}(k^{(0)},k^{(j)})e^{(k^{(0)}\phi^{(0)}+k^{(j)}\phi^{(j)})} \tag{34}$$

with j,

$$c^{(1)}(k^{(0)},k^{(j)})=\frac{i}{2(k^{(0)}+2^{-j}k^{(j)})}\left(1+\frac{1}{2^{2\tau}}\right) +\frac{2^{j}ik^{(0)}}{(2^{j}k^{(0)}+k^{(j)})^{2}}\left(\delta_{k^{(j)},2m}+\frac{1}{2^{2\tau}}\right) +\frac{2ik^{(0)}}{4(k^{(0)}+2^{-j}k^{(j)})^{2}} \times\sum_{\nu=0}^{j-2}2^{-\nu}\left(\delta_{k(j),2^{j-\nu}n}+\frac{1}{2^{2\tau}}\right), \tag{35}$$

where m and n are integers. We can see also that the two-dimensional projections of the invariant density will have an asymptotic form for large τ and that the convergence rate in this limit will also be proportional to $4^{-\tau}$.

In order to check if the arguments of Sec. II apply here, we should investigate the correlation function for small ϵ. A superficial inspection of Eq. (34) suggests that to leading nonvanishing order correlations between $\phi^{(0)}$ and $\phi^{(\tau)}$ decay as $2^{-\tau}$. That rate apparently differs from the convergence rate of the one particle density which, according to Eq. (33), is given by $4^{-\tau}$. Thus one has to clarify how the decay of correlations is related to the convergence properties for large delays. For that purpose let us have a look at perturbative result (27). If the delay term is Hölder continuous of order l then its Fourier coefficients decay like $G_k \sim k^{-l-1}$ and Eq. (27) tells us that the one particle density converges according to a power law $2^{-(l+1)\tau}$. If g is analytic then Fourier coefficients decay even faster, namely exponentially $G_k \sim \exp(-\alpha k)$ and the convergence of the one particle density is hyperexponential $\exp(-\alpha 2^{-\tau})$. Thus the mixing rate of the map together with the analytical properties of the delay term determine the convergence rate of the projected measure. In fact the correct convergence rate is reflected by a suitable correlation function, namely the pair correlation of $g(\phi)$ itself, which can be easily computed taking the Fourier series into account.

Although we have restricted our analysis to first order in ε we expect that higher order contributions will have the same qualitative behavior. In order to make this study more complete, we have reproduced the numerical analysis of Figs. 3 and 4 in the case of map (9). As it can be seen from the lower panel of Fig. 8, the one-dimensional projection converges also when ϵ is large. The convergence is smooth and the discrepancy between the measure at low and large delays depends on ϵ similarly as the first order approximation suggests.

IV. DISCUSSION

We have analyzed the limit of large delay in a particular time discrete system by an analytical perturbation expansion. The validity of the expansion has been confirmed by numerical simulations. Our result shows that projected measures converge in the limit of large delay, where the rate of convergence is determined by the mixing rate of the chaotic map and by analytic properties of the delay term. Chaos plays, of course, an important role for the convergence since otherwise correlations would not decay and smooth densities would not exist in general. However, it is not the plain mixing rate which is responsible for the rate of the convergence

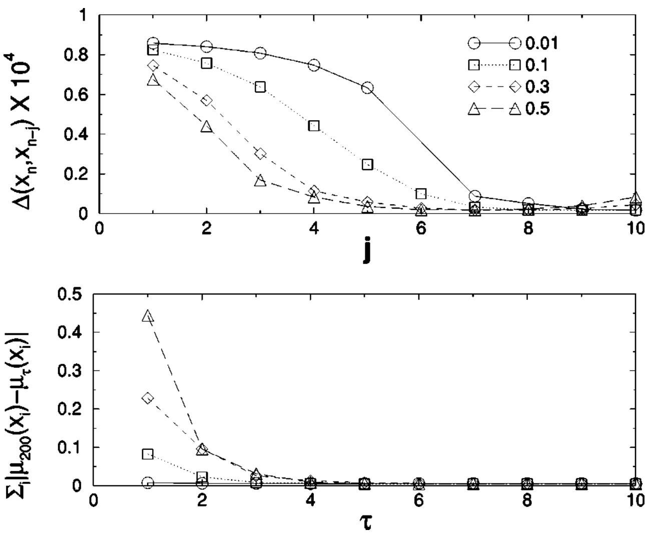

FIG. 8. $\Delta(x_n, x_{n-j})$ (upper panel) and the average difference between measures at low and large delay (lower panel) for the map (9).

but a particular pair correlation which involves the analytical properties of the delay term.

Our treatment was confined to a first order expansion. But we have good indications that our results are valid beyond such an order and we suppose that one might even be able to perform a formal proof of our statements along these lines. In addition, using, e.g., diagrammatic or projection techniques it should be possible to go beyond our simple perturbative treatment. But such advanced approaches go far beyond the scope of this present contribution. Analytic approaches so far are restricted to hyperbolic systems, i.e., essentially to maps on the torus. Maps on intervals, e.g., even the simple Bernoulli shift (3) lacks such a treatment until today. Nevertheless, numerical simulations indicate that qualitatively such models behave similar.

We have focused here on the scalar density $\rho(x)$ of Eq. (2) but other projections of the probability densities in spaces with smaller dimensions than the attractor itself are also observed to have an asymptotic form in the limit of large delay. One of these projections, the one on the manifold spanned by $\{x_{n+1}, x_n, x_{n-\tau}\}$, is relevant when phase space reconstruction methods are used to identify the dynamics from a time series [18,19]. From this projection one can also perform estimates of the metric entropy without using the minimal dimension required by the Takens theorem. Therefore, the study of other projections of the invariant density of a delayed system is an intersting issue for future investigations.

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